arXiv:1107.5200v1 [math.CA] 26 Jul 2011

JACOB'S LADDERS AND THE THREE-POINTS INTERACTION OF THE RIEMANN ZETA-FUNCTION WITH ITSELF

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Abstract. It is proved that some set of the values of $|\zeta(\sigma_0+i\varphi_1(t))|^2$ on every fixed line $\sigma=\sigma_0>1$ generates a corresponding set of the values of $|\zeta(\frac{1}{2}+it)|^2$ on the critical line $\sigma=\frac{1}{2}$ (i.e. we have an analogue of the Faraday law).

1. The result

1.1. Let us remind the theorem of H. Bohr (see [5], p. 253, Theorem 11.6(C)): The function $\zeta(s)$ attains every value $a\in\mathbb{C}$ except 0 infinitely many times in the strip $a<\sigma<1+\delta$. At the same time, this theorem does not allow us to prove anything about the set of the roots of the equation

$$|\zeta(\sigma_0+it)|=|a|,\ \sigma_0\in(1,1+\delta),$$

i.e. about the roots lying on every fixed line $\sigma=\sigma_0$.

In this paper we will study more complicated nonlinear equation

$$\left|\zeta\left(\frac{1}{2}+iu\right)\right|^2\left|\zeta\left(\frac{1}{2}+iv\right)\right|^2=\frac{1}{2}\zeta(2\sigma)\ln u,\ u,v>0,\ \sigma\geq\alpha>1 \tag{1.1}$$

where α is an arbitrary fixed value.

Remark 1. The theory of H. Bohr is not applicable on the equation (1.1). However, by the method of Jacob's ladders we obtain some information about the set of approximative solutions of this equation.

1.2. Let us remind that

$$\tilde{Z}^2(t)=\frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t},\ \varphi_1(t)=\frac{1}{2}\varphi(t) \tag{1.2}$$

where

$$\tilde{Z}^2(t)=\frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]}=\frac{Z^2(t)}{\left\{1+\mathcal{O}\left(\frac{\ln\ln t}{\ln t}\right)\right\}\ln t}, \tag{1.3}$$

(see [1], (3.9); [2], (1.3); [4], (1.1), (3.1)), and $\varphi(t)$ is the solution of the nonlinear integral equation

$$\int_0^{\mu[x(T)]}Z^2(t)e^{-\frac{2}{x(T)}t}\mathrm{d}t=\int_0^T Z^2(t)\mathrm{d}t.$$

Key words and phrases. Riemann zeta-function.

1.3.

Definition. If there are some sequences $\{u_n(\sigma_0)\}_{n=0}^\infty,\ \{v_n(\sigma_0)\}_{n=0}^\infty,\ \sigma_0>1$ for which the following condition are fulfilled

$$\text{(A)}\qquad \lim_{n\to\infty} u_n(\sigma_0)=\infty,\ \lim_{n\to\infty} v_n=\infty,$$

$$\text{(B)}\qquad \frac{\left|\zeta\left(\frac12+iu_n(\sigma_0)\right)\right|^2\left|\zeta\left(\sigma_0+iv_n(\sigma_0)\right)\right|^2}{\zeta(2\sigma_0)\ln u_n(\sigma_0)}=\frac12+o(1),$$

then we call each ordered pair $[u_n(\sigma_0),v_n(\sigma_0)]$ an *asymptotically approximate solution* (AA solution) of the nonlinear equation (1.1).

The following theorem holds true.

Theorem. Let us define the continuum set of sequences

$$\{K_n(T)\}_{n=0}^\infty,\ K_n\geq T_0[\varphi_1]$$

as follows

$$\text{(1.4)}\qquad \begin{aligned} &K_0=T,\ K_1=K_0+K_0^{1/3+2\epsilon},\ K_2=K_1+K_1^{1/3+2\epsilon},\dots, \\ &K_{n+1}=K_n+K_n^{1/3+2\epsilon},\dots\ . \end{aligned}$$

Then for every $\sigma_0:\ \ \sigma_0\geq\alpha>1$ there is a sequence $\{u_n(\sigma_0)\}_{n=0}^\infty,\ u_n(\sigma_0)\in(K_n,K_{n+1})$ such that

$$\text{(1.5)}\qquad \frac{\left|\zeta\left(\frac12+iu_n(\sigma_0)\right)\right|^2\left|\zeta\left(\sigma_0+i\varphi_1[u_n(\sigma_0)]\right)\right|^2}{\zeta(2\sigma_0)\ln u_n(\sigma_0)}=\frac12+\mathcal{O}\left(\frac{\ln\ln K_n}{\ln K_n}\right).$$

holds true where $\varphi_1[u_n(\sigma_0)]\in(\varphi_1(K_n),\varphi_1(K_{n+1}))$ and

$$\text{(1.6)}\qquad \rho\{[K_n,K_{n+1}];[\varphi_1(K_n),\varphi_1(K_{n+1})]\}\sim(1-c)\pi(K_n)\to\infty$$

as $n\to\infty$ where ρ denotes the distance of the segments, c is the Euler's constant and $\pi(t)$ is the prime-counting function, i.e. the ordered pair

$$[u_n(\sigma_0),\varphi_1[u_n(\sigma_0)]];\ \varphi_1[u_n(\sigma_0)]=v_n(\sigma_0)$$

is the AA solution of the nonlinear equation (1.1).

Remark 2. Let us point out that the formula (1.5) binds together the values of $\zeta(s)$ at three distinct points: at the point $\frac12+iu_n(\sigma_0)$ lying on the critical line $\sigma=\frac12$, and at the points $\sigma_0+i\varphi_1[u_n(\sigma_0)],2\sigma_0$ lying in the semi-plane $\sigma\geq\alpha>1$.

Remark 3. Since by the eq. (1.5) we have

$$\text{(1.7)}\qquad \left|\zeta\left(\frac12+iu_n(\sigma_0)\right)\right|^2\sim\frac{\zeta(2\sigma_0)\ln u_n(\sigma_0)}{2}\frac{1}{\left|\zeta\left(\sigma_0+i\varphi_1[u_n(\sigma_0)]\right)\right|^2},$$

$$\text{(1.8)}\qquad \left|\zeta\left(\sigma_0+i\varphi_1[u_n(\sigma_0)]\right)\right|^2\sim\frac{\zeta(2\sigma_0)\ln u_n(\sigma_0)}{2}\frac{1}{\left|\zeta\left(\frac12+iu_n(\sigma_0)\right)\right|^2},$$

then for the two parallel conductors placed at the positions of the lines $\sigma=\frac12$ and $\sigma=\sigma_0\geq\alpha>1$ some art of the Faraday law holds true: the sequence of the energies $\{|\zeta(\sigma_0+i\varphi_1[u_n(\sigma_0)])|^2\}_{n=0}^\infty$ on $\sigma=\sigma_0$ generates the sequence of the energies $\{|\zeta(\frac12+iu_n(\sigma_0))|^2\}_{n=0}^\infty$ on $\sigma=\frac12$, and vice versa (see (1.7), (1.8); the energy is proportional to the square of the amplitude of oscillations).

2. The local mean-value theorem

2.1. Let us remind that there is the *global* mean-value theorem (see [5], p. 116)

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\zeta(\sigma+it)|^2\mathrm{d}t=\zeta(2\sigma),\ \sigma>1.$$

However, for our purpose, we need the *local* mean-value theorem, i.e. the formula for the the integral

$$\int_T^{T+U}|\zeta(\sigma+it)|^2\mathrm{d}t,\ \sigma>1.$$

In this direction, the following lemma holds true.

Lemma. The formula

$$\int_T^{T+U}|\zeta(\sigma+it)|^2\mathrm{d}t=\zeta(2\sigma)U+\mathcal{O}(1) \tag{2.1}$$

holds true uniformly for $T,U>0,\ \sigma\geq\alpha$, where $\alpha>1$ is an arbitrary fixed value. The $\mathcal{O}$-constant depends of course on the choice of α.

Remark 4. The formula (2.1) is the asymptotic formula for $U\geq\ln\ln T$, for example.

2.2. **Proof of the Lemma.** Following the formula

$$\zeta(s)=\zeta(\sigma+it)=\sum_{n=1}^\infty\frac{1}{n^{\sigma+it}},\ \sigma>1$$

we obtain

$$|\zeta(\sigma+it)|^2=\zeta(2\sigma)+\sum_n\sum_{m\neq n}\frac{1}{(mn)^\sigma}\cos\left(t\ln\frac{n}{m}\right),$$

and

$$\begin{aligned}\int_T^{T+U}|\zeta(\sigma+it)|^2\mathrm{d}t&=\zeta(2\sigma)U+\mathcal{O}\left(\sum_n\sum_{m<n}\frac{1}{(mn)^\sigma\ln\frac{n}{m}}\right)=\\&=\zeta(2\sigma)U+S(m<n)\end{aligned} \tag{2.2}$$

uniformly for $T,U>0$. Let

$$S(m<n)=S\left(m<\frac{n}{2}\right)+S\left(\frac{n}{2}\leq m<n\right)=S_1+S_2. \tag{2.3}$$

Since we have $2<\frac{n}{m}$ in S_1, and

$$\sum_{n=1}^\infty\frac{1}{n^\sigma}=1+\sum_{n=2}^\infty\frac{1}{n^\sigma}<1+\int_1^\infty x^{-\sigma}\mathrm{d}x=1+\frac{1}{\sigma-1}$$

then

$$S_1=\mathcal{O}\left(\sum_n\sum_{m<\frac{n}{2}}\frac{1}{(mn)^\sigma}\right)=\mathcal{O}\left\{\left(\sum_{n=1}^\infty\frac{1}{n^\sigma}\right)^2\right\}=\mathcal{O}(1). \tag{2.4}$$

We put $m=n-r>\frac{n}{2};\ 1\leq r<\frac{n}{2}$ in S_2 and we obtain as usual

$$\ln\frac{n}{m}=\ln\frac{n}{n-r}=-\ln\left(1-\frac{r}{n}\right)>\frac{r}{n}.$$

Next, we have

$$\sum_{n=2}^\infty \frac{\ln n}{n^{2\sigma-1}}=\frac{\ln 2}{2^{2\sigma-1}}+\sum_{n=3}^\infty\frac{\ln n}{n^{2\sigma-1}}<\frac{\ln 2}{2^{2\sigma-1}}+\int_2^\infty \frac{\ln x}{x^{2\sigma-1}}\mathrm{d}x= \\ =\frac{\ln 2}{2^{2\sigma-1}}+\frac{2^{-2\sigma+1}}{\sigma-1}\ln 2+\frac{2^{-2\sigma}}{(\sigma-1)^2}=\mathcal{O}(1);\ 2^{-\sigma}\in\left(0,\frac 12\right).$$

Then

$$S_2=\mathcal{O}\left(\sum_n\sum_{r=1}^{n/2}\frac{n}{(mn)^\sigma r}\right)=\mathcal{O}\left(2^\sigma\sum_{n=2}^\infty\frac{\ln n}{n^{2\sigma-1}}\right)=\mathcal{O}(1). \tag{2.5}$$

Finally, from (2.2) by (2.3)-(2.5) the formula (2.1) follows.

3. Proof of the Theorem

3.1. Let us remind that the following lemma holds true (see [3], (2.5);[4], (3.3)): for every integrable function (in the Lebesgue sense) $f(x),\ x\in[\varphi_1(T),\varphi_1(T+U)]$ the following is true

$$\int_T^{T+U} f[\varphi_1(t)]\tilde{Z}^2(t)\mathrm{d}t=\int_{\varphi_1(T)}^{\varphi_1(T+U)} f(x)\mathrm{d}x,\ U\in\left(0,\frac{T}{\ln T}\right], \tag{3.1}$$

where $t-\varphi_1(t)\sim(1-c)\pi(t)$.

3.2. In the case $f(t)=|\zeta(\sigma_0+it)|^2,\ U=U_0=T^{1/3+2\epsilon}$ we obtain from (3.1) the following formula

$$\int_T^{T+U_0}|\zeta(\sigma_0+i\varphi_1(t))|^2\tilde{Z}^2(t)\mathrm{d}t=\int_{\varphi_1(T)}^{\varphi_1(T+U_0)}|\zeta(\sigma_0+it)|^2\mathrm{d}t. \tag{3.2}$$

Since (see (2.1))

$$\int_{\varphi_1(T)}^{\varphi_1(T+U_0)}|\zeta(\sigma_0+it)|^2\mathrm{d}t=\zeta(2\sigma_0)\{\varphi_1(T+U_0)-\varphi_1(T)\}+\mathcal{O}(1)= \\ =\frac 12\zeta(2\sigma_0)U_0\tan[\alpha(T,U_0)]+\mathcal{O}(1) \tag{3.3}$$

where (see (1.2))

$$\frac{\varphi_1(T+U_0)-\varphi_1(T)}{U_0}=\frac 12\frac{\varphi(T+U_0)-\varphi(T)}{U_0}=\frac 12\tan[\alpha(T,U_0)],$$

and (see [3], (2.6))

$$\tan[\alpha(T,U_0)]=1+\mathcal{O}\left(\frac{1}{\ln T}\right),$$

then (see (3.3))

$$\int_{\varphi_1(T)}^{\varphi_1(T+U_0)}|\zeta(\sigma_0+it)|^2\mathrm{d}t=\frac 12\zeta(2\sigma_0)U_0\left\{1+\mathcal{O}\left(\frac{1}{\ln T}\right)\right\}. \tag{3.4}$$

3.3. Next, by the first application of the mean-value theorem we obtain (see (1.3), (3.2))

$$\begin{aligned} & \int_T^{T+U_0} |\zeta(\sigma_0+i\varphi_1(t))|^2\tilde{Z}^2(t)\mathrm{d}t = \\ (3.5)\qquad & = \frac{1}{\left\{1+\mathcal{O}\left(\frac{\ln\ln\xi_1}{\ln\xi_1}\right)\right\}\ln\xi_1}\int_T^{T+U_0}|\zeta(\sigma_0+i\varphi_1(t))|^2\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t, \\ & \xi_1=\xi_1(\sigma_0;T,U_0)\in(T,T+U_0), \end{aligned}$$

and by the second application of this we obtain

$$\begin{aligned} & \int_T^{T+U_0}|\zeta(\sigma_0+i\varphi_1(t))|^2\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t = \\ (3.6)\qquad & = |\zeta(\sigma_0+i\varphi_1(\xi_2))|^2\left|\zeta\left(\frac{1}{2}+i\xi_2\right)\right|^2 U_0, \\ & \xi_2=\xi_2(\sigma_0;T,U_0)\in(T,T+U_0),\ \varphi_1(\xi_2)\in(\varphi_1(T),\varphi_1(T+U)); \\ & \ln\xi_1\sim\ln\xi_2. \end{aligned}$$

Hence, from (3.5) by (3.6) we have

$$\int_T^{T+U_0}|\zeta(\sigma_0+i\varphi_1(t))|^2\tilde{Z}^2(t)\mathrm{d}t = \frac{|\zeta(\sigma_0+i\varphi_1(\xi_2))|^2\left|\zeta\left(\frac{1}{2}+i\xi_2\right)\right|^2}{\left\{1+\mathcal{O}\left(\frac{\ln\ln\xi_2}{\ln\xi_2}\right)\right\}\ln\xi_2}U_0, \tag{3.7}$$

and from (3.2) by (3.4), (3.7) we obtain

$$\frac{|\zeta(\sigma_0+i\varphi_1(\xi_2))|^2\left|\zeta\left(\frac{1}{2}+i\xi_2\right)\right|^2}{\zeta(2\sigma_0)\ln\xi_2} = \frac{1}{2}+\mathcal{O}\left(\frac{\ln\ln\xi_2}{\ln\xi_2}\right). \tag{3.8}$$

3.4. Now, if we apply (3.8) in the case (see (1.4))

$$\begin{aligned} & [T,T+U_0]\to[K_n,K_{n+1}];\ \xi_2(\sigma_0)\to\xi_{2,n}(\sigma_0)\in(K_n,K_{n+1}); \\ & \xi_{2,n}(\sigma_0)=u_n(\sigma_0), \end{aligned}$$

then we obtain (1.5). The statement (1.7) follows from $t-\varphi_1(t)\sim(1-c)\pi(t)$.

4. Concluding remarks

Let us remind that in [1] we have shown the following formula holds true

$$\int_0^T Z^2(t)\mathrm{d}t = \varphi_1(T)\ln\varphi_1(T)+(c-\ln 2\pi)\varphi_1(T)+c_0+\mathcal{O}\left(\frac{\ln T}{T}\right),\ \varphi_1(T)=\frac{\varphi(T)}{2},$$

where $\varphi_1(T)$ is the Jacob's ladder. It is clear that $\varphi_1(T)$ is the asymptotic solution of the nonlinear transcendental equation

$$\int_0^T Z^2(t)\mathrm{d}t = V(T)\ln V(T)+(c-\ln 2\pi)V(T).$$

References

[1] J. Moser, 'Jacob's ladders and the almost exact asymptotic representation of the Hardy-Littlewood integral', Math. Notes 2010, **88**, pp. 414-422, arXiv: 0901.3973.

[2] J. Moser, 'Jacob's ladders and the multiplicative asymptotic formula for short and microscopic parts of the Hardy-Littlewood integral', (2009), arXiv:0907.0301.

[3] J. Moser, 'Jacob's ladders and the first asymptotic formula for the expression of the fifth order $Z[\varphi(t)/2+\rho_1]Z[\varphi(t)/2+\rho_2]Z[\varphi(t)/2+\rho_3]\hat{Z}^2(t)$ for the collection of disconnected sets', (2009), arXiv:0912.0130.

[4] J. Moser, 'Jacob's ladders, the iterations of Jacob's ladder $\varphi_1^k(t)$ and asymptotic formulae for the integrals of the products $Z^2[\varphi_1^n(t)]Z^2[\varphi^{n-1}(t)]\cdots Z^2[\varphi_1^0(t)]$ for arbitrary fixed $n\in\mathbb{N}$' (2010), arXiv:1001.1632.

[5] E.C. Titchmarsh, 'The theory of the Riemann zeta-function' Clarendon Press, Oxford, 1951.

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